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Extremes of Realizations of Continuous Time Stationary Stochastic Processes on Closed Intervals

OVE DITLEVSEN

Danmarks Ingeniørakademi, Rigensgade 11, Copenhagen, Denmark

Submitted by Richard Bellman

1. INTRODUCTION

In many engineering problems it is of great interest to know the distribution of the extremes in finite realizations of continuous time, stationary processes. From a mathematical point of view, it is extraordinarily difficult to give a rigorous and systematic theory for this subject. The asymptotic behavior of the extremes for some special processes is known, but only few results exist concerning the extremes of the sample function within a closed interval on the time axis (Darling and Siegert [4], Slepian [6]), except for the rather special and strange process where the values of the sample function at different times are complete independent. As all engineering problems are restricted to finite time or space intervals, it would be worth-while to try to get even rather crude results.

As an example we may consider a test bar of a ductile material. When an increasing tension force is applied to such a bar, it will start to yield in the weakest cross-section, when the force reaches some level. This level defines the yield strength of the bar. Now it is obvious that the yield force in the weakest cross-section is a minimum value of a finite sample function of a stochastic process, which in many cases may be assumed to be stationary. Thus, we just are concerned with the problem of finding the distribution of an extreme of a sample function defined on a closed interval of length as the length of the test bar.

2. THE CONDITION FUNCTION

Consider a real, separable, strict sense stationary stochastic process on the real t-axis. Then the process have such properties that the sample function $\xi(t)$ with probability 1 has a finite minimal value in every closed interval on the t-axis. The probability that the sample function $\xi(t)$ is greater than k everywhere in the closed interval $[a, a + x]$ is

$$f(x) = P(\min_{t \in [a, a+x]} \xi(t) > k). \tag{2.1}$$

The probability that $\xi(t)$ is greater than k whenever $t \in [a, a + x]$ under the condition that $\xi(t)$ is greater than k whenever $t \in [a - y, a]$, or conversely, is $f(x \mid y)$ and $f(y \mid x)$, respectively. It thus follows that

$$f(x + y) = f(x) f(y \mid x) = f(y) f(x \mid y). \tag{2.2}$$

If we introduce a *condition function* $C(x, y)$ defined by

$$f(x \mid y) = C(x, y) f(x), \qquad f(y \mid x) = C(y, x) f(y) \tag{2.3}$$

(2.1) can be written

$$f(x + y) = f(x) f(y)\, C(y, x) = f(y) f(x)\, C(x, y). \tag{2.4}$$

Thus the condition function is symmetric in x and y i.e.

$$C(x, y) = C(y, x). \tag{2.5}$$

Assume that $f(x)$ is a continuous function of $x \geqslant 0$. With $y = \Delta x$, (2.4) gives

$$f(x + \Delta x) = f(x) f(\Delta x)\, C(x, \Delta x), \tag{2.6}$$

which for $\Delta x \to 0$ gives

$$C(x, 0) = \frac{1}{f(0)}, \tag{2.7}$$

provided that $f(0) \neq 0$.

If the minima over disjoint intervals are mutually independent then $C(x, y) = 1$. This independence is often assumed in the applications, e.g., in the statistical theory of fracture in materials, see Weibull [8]. Even when x and y in (2.4) approach zero this assumption is preserved. From (2.7) it is seen that $f(0) = 1$, and not dependend of k, an obviously meaningless result in physical world.

3. Solution of the Functional Equation

The expression (2.3) in the form

$$\frac{f(x + y)}{f(x) f(y)} = C(x, y) \tag{3.1}$$

is a *functional equation* from which $f(x)$ can be determined when $C(x, y)$ is given. We seek solutions of (3.1) which are in $C^1[0, \infty[$. With $a \leqslant x$ we have

$$f(x) = f(a) f(x-a)\, C(a, x-a). \tag{3.2}$$

Partial differentiation of (3.2) with respect to x leads to

$$\frac{\partial f(x)}{\partial x} = f(a) f(x-a) \left(\frac{\partial C}{\partial y}\right)_{(a, x-a)} + f(a) \frac{\partial f(x-a)}{\partial(x-a)}\, C(a, x-a), \tag{3.3}$$

and if a converges to x (3.3) gives the differential equation

$$\frac{\partial f(x)}{\partial x} = f(x) \left[f(0) \left(\frac{\partial C}{\partial y}\right)_{(x,0)} + \left(\frac{\partial f(x)}{\partial x}\right)_{x=0} C(x, 0) \right]. \tag{3.4}$$

The complete integral to (3.4) determines all continuous differentiable solutions to the functional equation (3.1).[1]

By using (2.7), the relevant particular integral is for $f(0) \neq 0$

$$f(x) = f(0) \exp\left(f(0) \int_0^x \left(\frac{\partial C(x, y)}{\partial y}\right)_{(t,0)} dt + \frac{f'(0)}{f(0)}\, x \right). \tag{3.5}$$

We assume $0 < f(0) < 1$ and introduce the notation

$$g(x) = \left(\frac{\partial C(x, y)}{\partial x}\right)_{(x,0)} \frac{f(0)}{\ln f(0)} \tag{3.6}$$

$$c = -f'(0) \tag{3.7}$$

$$G(x) = \int_0^x g(t)\, dt \tag{3.8}$$

$$\lambda(x) = G(x) + 1 - \frac{cx}{f(0) \ln f(0)} \tag{3.9}$$

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[1] It must be emphasized that (3.4) only gives all solutions $f(x) \in C^1[0, \infty[$ (in fact it is only required that $df(x)/dx$ is continuous for $x = 0$). The gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

for instance is connected with the beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

through the functional equation

$$B(x, y) = \frac{\Gamma(x)\, \Gamma(y)}{\Gamma(x+y)}.$$

The gamma function is, however, not a solution to (3.4). This is due to the fact that $\Gamma(x)$ is not defined for $x = 0$.

Besides, nondifferentiable solutions are without interest for the problem in this paper.

such that

$$f(x) = f(0)^{\lambda(x)}. \tag{3.10}$$

Both $g(x) \in C[0, \infty[$ and c are *arbitrary*, but the interpretation of $f(x)$ as a probability limits the choice of $g(x)$ and c, (see below).

After this the functional equation (3.1), by substitution of (3.10), gives the condition function

$$C(x, y) = f(0)^{\lambda(x+y)-\lambda(x)-\lambda(y)} = f(0)^{G(x+y)-G(x)-G(y)-1}, \tag{3.11}$$

which is the only possibility if (3.1) is required to have solutions in $C^1[0, \infty[$.

From the differential equation (3.4), we get

$$g(0) = 0. \tag{3.12}$$

Since for $\tau \geqslant 0$

$$P(\min_{t\in[a,a+x]} \xi(t) > k) \geqslant P(\min_{t\in[a,a+x+\tau]} \xi(t) > k), \tag{3.13}$$

we have

$$f'(x) = f(0)^{\lambda(x)} \ln f(0) \frac{d\lambda(x)}{dx} \leqslant 0 \tag{3.14}$$

or

$$\frac{d\lambda(x)}{dx} = g(x) - \frac{c}{f(0) \ln f(0)} \geqslant 0, \tag{3.15}$$

which shows that

$$c \geqslant 0 \tag{3.16}$$

as $g(0) = 0$. As $\lambda(0) = 1$ we further have

$$\lambda(x) \geqslant 1. \tag{3.17}$$

THEOREM 1. *If a separable stochastic process with real sample functions $\xi(t)$, $-\infty < t < \infty$, has the properties:*

(1) *Stationarity in the strict sense.*

(2) $P(\min_{t\in[a,a+x]} \xi(t) > k) = f(x) \in C^1[0, \infty[$ *for a fixed level k.*

(3) $0 < f(0) < 1$.

Then the function $f(x)$ is of the form

$$f(x) = e^{-cx/f(0)} f(0)^{1+G(x)}, \tag{3.18}$$

where $c \geqslant 0$ and $f(0) = P(\xi(t) > k)$ are independent of x, while

$$G(x) = \int_0^x g(t)\, dt, \tag{3.19}$$

where $g(x) \in C[0, \infty[$ and $g(0) = 0$.

4. Some Dependence Assumptions

Let us assume the existence of stochastic processes which in addition to the properties of Theorem 1 have the following properties

4. The dependence is decreasing monotonously such that

$$\frac{\partial C(x, y)}{\partial y} < 0 \tag{4.1}$$

whenever $x \geqslant 0$ and $y > 0$.

5. The condition function $C(x, y)$ has for every $x \geqslant 0$ a limit

$$C(x, \infty) \geqslant 1 \tag{4.2}$$

for $y \to \infty$.

6. The minima in two consecutive intervals approach mutual independence as the length of the two intervals becomes large, i.e.,

$$C(x, y) \to 1 \tag{4.3}$$

for $x \to \infty$ and $y \to \infty$.

The properties 4-6 are consistent as $C(0, 0) = 1/f(0) > 1$ due to 3. Substituting (3.11) in the condition (4.1) gives

$$\frac{\partial C(x, y)}{\partial y} = [g(x + y) - g(y)]\, C(x, y) \ln f(0) < 0, \tag{4.4}$$

showing that

$$g(x + y) > g(y). \tag{4.5}$$

Thus $g(x)$ is monotonously increasing for $x \geqslant 0$, and $\lim_{x\to\infty} g(x) = g(\infty)$ exists, eventually with $g(\infty) = \infty$. As $g(0) = 0$ we have $g(x) \geqslant 0$ for $x \geqslant 0$.

We will show, by use of the properties 3 and 5, that $g(\infty) < \infty$. As $g(x) \in C[0, \infty[$ we have from (3.8) with $\Theta \in]0, 1[$

$$G(x + y) - G(y) = \left(\int_0^{x+y} - \int_0^{y}\right) g(t)\, dt = xg(y + \Theta x) \to xg(\infty) \tag{4.6}$$

for $y \to \infty$, and comparing this with

$$C(x, y) = f(0)^{-G(x)-1} f(0)^{G(x+y)-G(y)} \to C(x, \infty) \geqslant 1 \tag{4.7}$$

for $y \to \infty$, fulfilles the proof. With $\beta = g(\infty)$ (4.6) and (4.7) give the inequality

$$\beta x - G(x) - 1 = \int_0^x [\beta - g(t)]\, dt - 1 \leqslant 0 \tag{4.8}$$

as $0 < f(0) < 1$. From $\beta - g(x) > 0$ for $x \geqslant 0$ follows that the limit of the integral exists for $x \to \infty$, and from the property (4.3) we have the value

$$\int_0^\infty [\beta - g(t)]\, dt = 1. \tag{4.9}$$

Defining

$$A(x) = 1 - \int_0^x [\beta - g(t)]\, dt = 1 + G(x) - \beta x, \tag{4.10}$$

we may summarize in

LEMMA. *If the stochastic process of Theorem 1 possesses the properties 4, 5, and 6, then*

$$1 + G(x) = A(x) + \beta x, \tag{4.11}$$

where the function $A(x)$ has the properties

$$A(0) = 1, \qquad \lim_{x\to\infty} A(x) = 0 \tag{4.12}$$

$$A'(0) = -\beta, \qquad A'(x) < 0 \qquad \text{for} \qquad x \geqslant 0. \tag{4.13}$$

The function $A(x)$ and the constant β may depend on the level k.

In the following we will specialize our considerations to normal processes with the properties of Theorem 1.

5. THE CONSTANT c FOR NORMAL PROCESSES

Let $\xi(t)$, $-\infty < t < \infty$, be a real, separable, stationary normal process, $E\xi(t) = 0$, $E\xi^2(t) = 1$, with the correlation function

$$\rho(\tau) = E[\xi(t+\tau)\,\xi(t)] = \int_0^\infty \cos \lambda\tau\, h(\lambda)\, d\lambda, \tag{5.1}$$

where the spectral density function $h(\lambda)$ is of bounded variation in $[0, \infty[$ and satisfies the Hunt condition [5]

$$\int_0^\infty \lambda^2[\ln(1+\lambda)]^\alpha\, h(\lambda)\, d\lambda < \infty \tag{5.2}$$

for some $\alpha > 1$. Then $\xi(t) \in C^1]-\infty, \infty[$ with probability 1, while

$$\frac{d^2\rho(\tau)}{d\tau^2} = -\int_0^\infty \cos \lambda\tau\, \lambda^2 h(\lambda)\, d\lambda \in C]a, b[, \tag{5.3}$$

where $a \leqslant b$ are real numbers.

We consider the behavior of the sample function $\xi(t)$ in an interval of length τ, say, the interval $[0, \tau]$. Then

$$\begin{aligned} f(\tau) &= P(\min_{t\in[0,\tau]} \xi(t) > k) \\ &= P(\xi(0) > k, \xi(\tau) > k) - P(\xi(0) > k, \xi(\tau) > k, \min_{t\in[0,\tau]} \xi(t) < k) \\ &\qquad - P(\xi(0) > k, \xi(\tau) > k, \min_{t\in[0,\tau]} \xi(t) = k). \end{aligned} \tag{5.4}$$

Bulinskaya [1] has shown that the probability that $\xi(t)$ becomes tangent to the level k within a closed interval is zero. Thus the last term in (5.4) is zero.

We may evaluate the term $P(\xi(0) > k, \xi(\tau) > k, \min_{t\in[0,\tau]} \xi(t) < k)$ by use of a lemma proved by Dobrushin and published in a work of Volkonskii [7].

LEMMA. *If any two events do not occur simultaneously, and the intensity of the sequence is finite, then the sequence is ordinary.*

A sequence is called *ordinary* if the probability of the appearance of at least two events in $[0, t]$ is $0(t)$ for $t \to 0$. The *intensity* of the sequence is the mathematical expectation μ of the number of events during a unit of time.

Let the events be crossings of the level k by the sample function $\xi(t)$. Under the present conditions Bulinskaya [1] has shown that

$$\mu = E[N_k(1)] = \frac{1}{\pi} \sqrt{-\rho''(0)}\, e^{-k^2/2}, \tag{5.5}$$

where $N_k(\tau)$ is the number of crossings of the level k in an interval of length τ. Then we may apply the lemma to the evaluation.

$$P(\xi(0) > k, \xi(\tau) > k, \min_{t\in[0,\tau]} \xi(t) < k) < P(N_k(\tau) \geqslant 2) = o(\tau). \tag{5.6}$$

Now we can concentrate on the first term of (5.4).

We get with $\rho = \rho(\tau)$

$$\begin{aligned} f(\tau) + o(\tau) &= P(\xi(0) > k, \xi(\tau) > k) \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_k^\infty \int_k^\infty \exp\left(-\frac{1}{2}\frac{x^2 - 2\rho xy + y^2}{1-\rho^2}\right) dx\, dy \\ &= \left(\frac{1}{\sqrt{2\pi}} \int_k^\infty \exp\left(-\frac{1}{2} t^2\right) dt\right)^2 \\ &\qquad + \frac{1}{2\pi} \int_0^\rho \exp\left(-\frac{k^2}{1+t}\right) \frac{dt}{\sqrt{1-t^2}}. \end{aligned} \tag{5.7}$$

This identity has been given by Cramér [2], p. 514 line 4.

As

$$\lim_{\tau\to 0}\left(\frac{\rho'}{\sqrt{1-\rho}}\right)^2 = \lim_{\tau\to 0}\left(\frac{2\rho'\rho''}{-\rho'}\right) = -2\rho''(0), \tag{5.8}$$

according to l'Hospital's rule, ($\rho'(0) = 0$), differentiation of (5.7) gives

$$c = -\lim_{\tau\to 0}\frac{d}{d\tau}\left(f(\tau) + o(\tau)\right) = -f'(0) = \sqrt{-\frac{\rho''(0)}{2\pi}}\,\varphi(k), \tag{5.9}$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}\,. \tag{5.10}$$

It is seen that

$$c = \tfrac{1}{2}\,E[N_k(1)] = \tfrac{1}{2}\,\mu, \tag{5.11}$$

c.f. (5.5).

Writing $f(\tau) = f(\tau, k)$, where k is the level, and $p_n = P(N_k(\tau) = n)$, we have

$$P(N_k(\tau) = 0) = P(\min_{t\in[0,\tau]} \xi(t) > k) + P(\max_{t\in[0,\tau]} \xi(t) < k)$$

$$= f(\tau, k) + f(\tau, -k) = 1 - \sum_{n=1}^{\infty} p_n \tag{5.12}$$

due to the symmetry of a normal process. As

$$E[N_k(\tau)] = \tau E[N_k(1)] = \tau\mu = \sum_{n=1}^{\infty} np_n \tag{5.13}$$

and $f(0, k) + f(0, -k) = 1$, we may rewrite (5.12) as

$$-\frac{1}{\tau}\,[f(\tau, k) - f(0, k)] - \frac{1}{\tau}\,[f(\tau, -k) - f(0, -k)]$$

$$= \mu - \frac{1}{\tau}\sum_{n=2}^{\infty}(n-1)\,p_n\,, \tag{5.14}$$

which, with use of $\varphi(k) = \varphi(-k)$, for $\tau \to 0$ gives

$$2c = \mu - \lim_{\tau\to 0}\left(\frac{1}{\tau}\sum_{n=2}^{\infty}(n-1)\,p_n\right), \tag{5.15}$$

showing that for normal processes with the present properties

$$\sum_{n=2}^{\infty} (n-1)\, P(N_k(\tau) = n) = o(\tau) \tag{5.16}$$

as $2c = \mu$. This is a stronger assertion than that given by Dobrushin's lemma.

We now have

THEOREM 2. *If the process of Theorem 1 possesses the following properties:*

(1) *Normal* (0, 1) *with correlation function* $\rho(x)$.

(2) *Satisfies the Hunt condition (5.2).*

Then for a fixed level k

$$P(\min_{t\in[a,a+x]} \xi(t) > k) = \exp\left(-\frac{\varphi(k)}{1-\Phi(k)}\gamma x\right)(1-\Phi(k))^{1+G(x)}, \tag{5.17}$$

where

$$\Phi(k) = \int_{-\infty}^{k} \varphi(t)\, dt; \qquad \varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \tag{5.18}$$

and

$$\gamma = \sqrt{\frac{-\rho''(0)}{2\pi}}. \tag{5.19}$$

An assertion about the maximum of the sample function in a closed interval can immediately be obtained from (5.17)

$$P(\max_{t\in[a,a+x]} \xi(t) < k) = \exp\left(-\frac{\varphi(k)}{\Phi(k)}\gamma x\right)\Phi(k)^{1+G(x)} \tag{5.20}$$

for fixed k.

A linear transformation of the t-axis preserves the set of values which the sample function $\xi(t)$ takes on within a closed interval.

Thus the interval lengths may be measured relative to a characteristic length of the correlation function, say the abscissa T to the point of gravity, i.e.,

$$T = \frac{\int_0^\infty x\rho(x)\, dx}{\int_0^\infty \rho(x)\, dx}, \tag{5.21}$$

and we can replace x by x/T in all formulas in this section.

6. Asymptotic Behavior of the Extremes as the Interval Length Becomes Infinite

The main results of this paper are given in the theorems and the lemma. It would, however, also be interesting to investigate the asymptotic behavior of the extremes, as the interval length becomes infinite.

We will restrict ourselves to the normal process dealt with in the last section. In addition, we will assume that the process has the properties 4-6 such that the lemma may be used. Finally is assumed that

$$\lim_{k\to\infty} \frac{\beta(k)}{k} = 0. \tag{6.1}$$

Define for $x > 1$ a new random variable η by

$$\max_{t\in[a,a+x]} \xi(t) = \sqrt{2\ln x} + \frac{\eta}{\sqrt{2\ln x}}. \tag{6.2}$$

Then

$$P(\eta \leqslant y) = P\left(\max_{t\in[a,a+x]} \xi(t) \leqslant \sqrt{2\ln x} + \frac{y}{\sqrt{2\ln x}}\right). \tag{6.3}$$

We will first examine the factor $\exp(-\varphi(k)\gamma x/\Phi(k))$ in (5.20). With $u = \sqrt{2\ln x}$ we have

$$x\varphi\left(\sqrt{2\ln x} + \frac{y}{\sqrt{2\ln x}}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2}u^2 - \frac{1}{2}\left(u + \frac{y}{u}\right)^2\right) \to \frac{1}{\sqrt{2\pi}} e^{-y} \tag{6.4}$$

for $x \to \infty$. Then

$$\exp\left(-\frac{\varphi\left(\sqrt{2\ln x} + \frac{y}{\sqrt{2\ln x}}\right)\gamma x}{\Phi\left(\sqrt{2\ln x} + \frac{y}{\sqrt{2\ln x}}\right)}\right) \to \exp\left(-\frac{\gamma}{\sqrt{2\pi}} e^{-y}\right). \tag{6.5}$$

It is sufficient to consider the second term βx of the exponent in the second factor of (5.20), as $A(x) \to 0$ for $x \to \infty$, (see (4.16)). Writing $\beta = \beta(k)$, we have

$$x\beta\left(\sqrt{2\ln x} + \frac{y}{\sqrt{2\ln x}}\right) \ln \Phi\left(\sqrt{2\ln x} + \frac{y}{\sqrt{2\ln x}}\right)$$

$$= e^{u^2/2}\beta\left(u + \frac{y}{u}\right) \ln \Phi\left(u + \frac{y}{u}\right); \tag{6.6}$$

the limit of which, due to l'Hospital's rule, is the same as the limit of

$$\lim_{u\to\infty} \beta(u) \frac{\ln \Phi(u)}{e^{-\,{}^{2}/2}} = \lim_{u\to\infty} \frac{\beta(u)}{u} \lim_{u\to\infty} \left(- \frac{\varphi(u)}{\Phi(u)\; e^{-u^2/2}} \right)$$

$$= - \frac{1}{\sqrt{2\pi}} \lim_{u\to\infty} \frac{\beta(u)}{u} = 0 \tag{6.7}$$

due to the assumption (6.2).

The random variable

$$\zeta = \eta + \ln \frac{\sqrt{2\pi}}{\gamma}, \tag{6.8}$$

then, has the property

$$P(\zeta \leqslant z) = P\left(\eta \leqslant z - \ln \frac{\sqrt{2\pi}}{\gamma}\right) \to e^{-e^{-z}} \tag{6.9}$$

for $x \to \infty$.

In fact *Cramér* [3] *has, in quite another way, shown that the random variable* ζ *defined by*

$$\max_{t\in[a,a+x]} \xi(t) = \sqrt{2 \ln x} - \frac{\ln (2\pi/\sqrt{-\rho''(0)})}{\sqrt{2 \ln x}} + \frac{\zeta}{\sqrt{2 \ln x}} \tag{6.10}$$

under the conditions of Section 5 and the further condition

$$\int_0^\infty \lambda^4 h(\lambda)\, d\lambda < \infty, \tag{6.11}$$

in addition to a condition about strong mixing has the limiting distribution (6.9) as $x \to \infty$.

The condition (6.11) and the mixing condition is not necessary for the property

$$\max_{t\in[a,a+x]} \xi(t) - \sqrt{2 \ln x} \to 0 \tag{6.12}$$

in probability for $x \to \infty$, which was shown by Cramer in [2].

It has not been possible for the author to find a connection between the assumptions of Cramér and the assumptions of this paper.

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